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TRIANGULAR DECOMPOSITION OF A POSITIVE DEFINITE MATRIX PLUS A SYMMETRIC DYAD WITH APPLICATIONS TO KALMAN FILTERING

William S. Agee, et al

National Range Operations Directorate
White Sands Missile Range, New Mexico

October 1972

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BY

WILLIAM S. AGEE and ROBERT H. TURNER

OCTOBER 1972

MATHEMATICAL SERVICES BRANCH
ANALYSIS & COMPUTATION DIVISION
NATIONAL RANGE OPERATIONS DIRECTORATE
WHITE SANDS MISSILE RANGE, NEW MEXICO

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TRIANGULAR DECOMPOSITION OF A POSITIVE DEFINITE MATRIX PLUS A SYMMETRIC DYAD WITH APPLICATIONS TO KALMAN FILTERING

I. INTRODUCTION. Given a positive definite matrix P Choleski's theorem states that there exists a real, non-singular, lower triangular matrix L such that

$$LL^T = P \tag{1}$$

Furthermore if the diagonal elements of L are taken to be positive the decomposition is unique. L is called the square root of P. The Choleski algorithm for decomposition of P is presented in [1] and [2]. The Choleski decomposition is very useful in many numerical linear algebra problems. In particular it provides a useful numerical technique in the matrix square root formulation of the Kalman filter [3]. The triangular decomposition of Choleski is extended below to the decomposition of a positive definite matrix P plus a symmetric dyad cxx^T.

II. FIRST TRIANGULAR DECOMPOSITION ALGORITHM. Suppose we have a lower triangular decomposition L of a positive definite matrix P. The elements of L must satisfy the equations.

$$\sum_{j=1}^{i} \ell(i,j)\ell(k,j) = P_{ik} \quad k>i \tag{2}$$

$$\sum_{j=1}^{i} \ell^2(i,j) = P_{ii} \tag{3}$$

Consider the problem of computing the triangular decomposition L' of the modified matrix

$$P' = P + cxx^T \tag{4}$$

given the decomposition L of P. The decomposition L' must satisfy the equations

$$\sum_{j=1}^{i} \ell'(i,j)\ell'(k,j) = \sum_{j=1}^{i} \ell(i,j)\ell(k,j) + cx_i x_k, \; k>i \tag{5}$$

$$\sum_{j=1}^{i} \ell'^2(i,j) = \sum_{j=1}^{i} \ell^2(i,j) + cx_i^2 \tag{6}$$

First consider (5) and (6) for the case of $i=1$. In this case

$$\ell'(1,1)\ell'(k,1) = \ell(1,1)\ell(k,1) + cx_1 x_k \quad k>1 \tag{7}$$

$$\ell'^2(1,1) = \ell^2(1,1) + cx_1^2 \tag{8}$$

The first column of the modified matrix L' is easily computed from (7) and (8). Now rewrite (5) as

$$\sum_{j=2}^{i} \ell'(i,j)\ell'(k,j) + \ell'(i,1)\ell'(k,1) = \sum_{j=2}^{i} \ell(i,j)\ell(k,j) + \ell(i,1)\ell(k,1) + cx_i x_k \tag{9}$$

The second term on the left of (9) can be computed from (7) as

$$\ell'(i,1)\ell'(k,1) = \frac{\ell^2(1,1)}{\ell'^2(1,1)}\,\ell(i,1)\ell(k,1) + \frac{\ell(i,1)\ell(1,1)}{\ell'^2(1,1)}\,cx_1x_k + \tag{10}$$

$$\frac{\ell(k,1)\ell(1,1)}{\ell'^2(1,1)}\,cx_1x_i + \frac{c^2x_1^2}{\ell'^2(1,1)}\,x_ix_k$$

Substituting (10) into (9) and combining terms gives

$$\sum_{j=2}^{i} \ell'(i,j)\ell'(k,j) = \sum_{j=2}^{i} \ell(i,j)\ell(k,j) + c^{(1)}x_i^{(1)}x_k^{(1)} \tag{11}$$

where

$$c^{(1)} = \frac{c\ell^2(1,1)}{\ell'^2(1,1)} \tag{12}$$

$$x_j^{(1)} = \left(x_i - \frac{x_1\ell(j,1)}{\ell(1,1)}\right) \tag{13}$$

Similarly, (6) can be written as

$$\sum_{j=2}^{i} \ell'^2(i,j) + \ell'^2(i,1) = \sum_{j=2}^{i} \ell^2(i,j) + \ell^2(i,1) + cx_i^2 \tag{14}$$

Substituting from (7) for the second term on the left of (14) and combining terms gives

$$\sum_{j=2}^{i} \ell'^2(i,j) = \sum_{j=2}^{i} \ell^2(i,j) + c^{(1)} x_i^{(1)^2} \tag{15}$$

Equations (11) and (15) define a decomposition problem equivalent to the original problem defined by (5) and (6) but with the dimension reduced by one. It is easily seen that the application of the above technique n times, each time reducing the dimension of the decomposition problem by one, solves the original decomposition problem. The following equations summarize the algorithm for the triangular decomposition of $P+cxx^T$.

$$\ell'(i,i) = \left[\ell^2(i,i)+c^{(i)} x_i^{(i)^2} \right]^{1/2} \qquad i = 1,n \tag{16}$$

$$\ell'(k,i) = \frac{\ell(i,i)}{\ell'(i,i)}\, \ell(k,i) + \frac{c^{(i)} x_i^{(i)} x_k^{(i)}}{\ell'(i,i)} \qquad i + 1 \leq k \leq n \tag{17}$$

$$x_j^{(i+1)} = x_j^{(i)} - \frac{x_i^{(i)} \ell(j,i)}{\ell(i,i)} \qquad i + 1 \leq j \leq n \tag{18}$$

$$c^{(i+1)} = c^{(i)} \left(\frac{\ell(i,i)}{\ell'(i,i)} \right)^2 \tag{19}$$

III. <u>SECOND TRIANGULAR DECOMPOSITION ALGORITHM</u>. An alternate decomposition of positive definite matrix is possible. Let L be a unit lower triangular matrix, i.e., having ones along the diagonal. Then a positive definite matrix P can be written as

$$P = LDL^T \tag{20}$$

where D is a diagonal matrix of positive numbers. An algorithm for computing the decomposition

$$L'D'L'^T = LDL^T + cxx^T \tag{21}$$

can be derived in a manner parallel to the previous decomposition algorithm. The algorithm is as follows

$$d'_i = d_i + c^{(i)}x_i^{(i)^2} \qquad i = 1,n \tag{22}$$

$$x_k^{(i+1)} = x_k^{(i)} - x_i^{(i)}\ell(k,i) \tag{23}$$

$$\left.\begin{array}{l} \ell'(k,i) = \ell(k,i) + \dfrac{c^{(i)}x_i^{(i)}}{d'_i}\, x_k^{(i+1)} \end{array}\right\} \; i + 1 \leq k \leq n \tag{24}$$

$$c^{(i+1)} = c^{(i)}\left(\frac{d_i}{d'_i}\right)^2 \tag{25}$$

IV. <u>APPLICATION TO KALMAN FILTERING I.</u> At a measurement update in a discrete Kalman filter the state estimate is given by

$$\hat{x}_k = \hat{x}_{k/k-1} + P_k H_k^T R_k^{-1}\left(z_k - H_k\hat{x}_{k/k-1}\right) \tag{26}$$

where the covariance matrix P_k is

$$P_k = \left(P_{k/k-1}^{-1} + H_k^T R_k^{-1} H_k \right)^{-1} \tag{27}$$

Consider the measurement z_k to be a scalar since the vector measurement problem can always be reduced to this case, see [3]. In this scalar case H_k is a row vector, R_k^{-1} is a scalar, and P_k is n×n. In the matrix square root formulation of the Kalman filter given in [3], (27) is the inversion of a matrix plus a dyad which is simply expressed by the method of modification given in Householder [4]. This gives

$$P_k = P_{k/k-1} - \frac{P_{k/k-1} H_k^T H_k P_{k/k-1}}{R_k + H_k P_{k/k-1} H_k^T} \tag{28}$$

Since P_k and $P_{k/k-1}$ are positive definite write them as

$$P_k = L_k L_k^T$$

and

$$P_{k/k-1} = L_{k/k-1} L_{k/k-1}^T$$

Assume that $L_{k/k-1}$ is lower triangular with positive diagonal elements. Substituting in (28)

$$L_k L_k^T = L_{k/k-1} L_{k/k-1}^T - c_k L_{k/k-1} L_{k/k-1}^T H_k^T H_k L_{k/k-1} L_{k/k-1}^T \tag{29}$$

$$c_k = 1/\left(R_k + H_k P_{k/k-1} H_k^T \right) \tag{30}$$

Let

$$u_k = L_{k/k-1}^T H_k^T \tag{31}$$

and

$$w_k = L_{k/k-1} u_k \tag{32}$$

Then

$$L_k L_k^T = L_{k/k-1} L_{k/k-1}^T - c_k w_k w_k^T \tag{33}$$

Thus (33) is in the proper form for application of the first algorithm. In addition to the above computations the state estimate given by (26) must also be computed. Some manipulation shows that (26) can be written as

$$\hat{x}_k = \hat{x}_{k/k-1} + c_k w_k \left(z_k - H_k \hat{x}_{k/k-1} \right) \tag{34}$$

Now consider the numerical efficiency of the application. The execution of (31) and (32) takes n(n+1) (mult). (30) takes n (mult) and 1 (div), (34) takes (n+1) (mult). The decomposition algorithm for (33) requires

$$\left(3n^2+9n-6\right) /2 \text{ (mult)}$$

$2(n-1)$ (div) and n $\sqrt{\ }$'s. Thus the use of the first decomposition algorithm requires

$$\left(5/2\ n^2+15/2\ n-2\right)\ (\text{mult})$$

$2n-1$ (div) and n $\sqrt{\ }$'s. The technique presented in [3] which we have been using in our Kalman filter program requires about $3n^2+2n$ (mult) but does not generate a triangular square root matrix.

Now consider the numerical efficiency of the second algorithm. A development paralleling (29)-(34) gives

$$L_k D_k L_k^T = L_{k/k-1} D_{k/k-1} L_{k/k-1}^T - c_k w_k w_k^T \tag{35}$$

where

$$u_k = D_{k/k-1} L_{k/k-1}^T H_k^T \tag{36}$$

$$w_k = L_{k/k-1} u_k \tag{37}$$

$$c_k = 1/\left(R_k + H_k P_{k/k-1} H_k^T\right) \tag{38}$$

The use of the second algorithm and execution of (34)-(38) requires $2n^2+7n$ (mult) and n (div).

The application of the second algorithm results in fewer operations than either our present algorithm or the first algorithm presented above plus the benefit of having a covariance square root which is triangular.

V. APPLICATION TO KALMAN FILTERING II. Rather than compute the square root of the covariance matrix the square root of the inverse covariance matrix may be computed. The updating of the inverse covariance matrix is given by (27).

$$P_k^{-1} = P_{k/k-1}^{-1} + H_k^T R_k^{-1} H_k \tag{39}$$

since the inverse covariance is also positive definite, write

$$P_{k/k-1}^{-1} = L_{k/k-1}^{-1} \, D_{k/k-1}^{-1} \, L_{k/k-1}^{-T}$$

and

$$P_k^{-1} = L_k^{-1} D_k^{-1} L_k^{-T}$$

Again considering scalar measurements let $u_k = H_k^T$ and $c_k = 1/R_k$. Then

$$L_k^{-1} D_k^{-1} L_k^{-T} = L_{k/k-1}^{-1} D_{k/k-1}^{-1} L_{k/k-1}^{-T} + c_k u_k u_k^T \tag{40}$$

which is in the form for application of the second algorithm. The computation of the corrected state estimate given by (26) requires the computation of

$$y_k = \frac{P_k H_k^T}{R_k} \left(z_k - H_k \hat{x}_{k/k-1} \right) \tag{41}$$

Let $e_k = c_k \left(z_k - H_k \hat{x}_{k/k-1} \right) u_k$ and rewrite (41) as

$$P_k^{-1} y_k = L_k^{-1} D_k L_k^{-T} y_k = e_k \tag{42}$$

The solution of (42) for y_k requires the solution of two sets of linear equations each with a triangular coefficient matrix. The solution of a triangular set of linear equations is a standard procedure in numerical analysis. The solution of (42) for the vector y_k requires about n^2+n (mult) and n (div). The solution of the Kalman filter equations (40) and (42) at a measurement update requires about n^2+5n (mult) and n (div) using the second decomposition algorithm. The specification of (40) and (42) as the measurement update equations requires that $L_{k/k-1}^{-1}$ be computed at a time update. In the special case where there are no process dynamics, i.e. we are estimating constant parameters, and there is no process noise $L_{k/k-1}^{-1}=L_{k-1}^{-1}$ so that no additional computation is required to obtain $L_{k/k-1}^{-1}$. Although this is a special case, it is of interest in the bias filter portion of the WSMR BET program where the measurement biases are assumed to be constant in time.

VI. APPLICATION TO KALMAN FILTERING III. The use of the second triangular decomposition algorithm in the WSMR BET, which is an extended Kalman filter, has resulted in a significant increase in numerical efficiency, however, the motivating factor for the development and use of this algorithm was for the application described below.

The Kalman filter in the WSMR BET is divided into two filters, the zero-bias filter which produces trajectory state estimates x_k^* and the

bias filter which produces estimates $\hat{b}_k$ of the measurement biases. These two estimates are combined to form the optimal trajectory state estimate $\hat{x}_k$ defined by

$$\hat{x}_k = x_k^* + T_k \hat{b}_k \tag{43}$$

where T_k is a combining matrix which must satisfy equations determined from the orthogonality properties of the state estimates. One property of the filter decomposition given in (43) is the requirement that x_k^* and $\hat{b}_k$ be orthogonal. This requirement is naturally satisfied by the equations governing x_k^*, $\hat{b}_k$, and T_k except at a point where a measuring instrument is deleted from the filtering process. An instrument may be dropped because it is no longer taking observations, its bias is too large, or the measurements are chronically inconsistent with their statistics. Let x_-^*, P_-^* denote the zero-bias state estimate and its covariance just prior to dropping a measurement from the filtering solution and let x_+^* and P_+^* be the same quantities immediately after dropping the measurement. Similarly let $\hat{b}_-$ and $\hat{b}_+$ denote the bias state estimates before and after dropping a measurement. $\hat{b}_+$ is formed by deleting the component of $\hat{b}_-$ corresponding to the measurement being dropped. T_- and T_+ are the combining matrices before and after. T_+ has one less column than T_-. P_{b-} and P_{b+} are the bias covariance matrices before and after dropping a measurement. P_{b+} is formed from P_{b-} by deleting the row and column corresponding to the measurement being dropped. The updating equation for x^* is

$$x_+^* = x_-^* + t_i\left(\hat{b}_i - \ell^T \hat{b}_+\right) \tag{44}$$

where t_i is the column of T_- being deleted, $\hat{b}_i$ is the bias estimate for the measurement being dropped, and ℓ is vector chosen so that x_+^* and $\hat{b}_+$ will be orthogonal in the usual statistical sense. The updating equation for P^* is

$$P_+^* = P_-^{*'} + t_i t_i^T \left(P_{b-}(i,i) - \ell^T P_{b+} \ell \right) \tag{45}$$

If

$$P_-^* = C_-^* D_-^* C_-^{*T}$$

and

$$P_+^* = C_+^* D_+^* C_+^{*T}$$

then (45) is in a form for application of the second triangular decomposition algorithm. In addition, let

$$P_{b-} = C_{b-} D_{b-} C_{b-1}^T$$

and

$$P_{b+} = C_{b+} D_{b+} C_{b+}^T$$

Let the i^{th} row and column C_{b-} be deleted. Also let the i^{th} row and column of D_{b-} be deleted. Call the resulting matrices C_{b-}' and D_{b-}'.

Then

$$C_{b+}D_{b+}C_{b+}^T = C_{b-}^{*}D_{b-}^{*}C_{i}^{*} + D_i c_i c_i^T \tag{46}$$

where c_i is the column deleted from C_{b-} and d_i is the diagonal element deleted from D_{b-}. Thus (46) presents another application for the decomposition algorithm.

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